MATH 245 S18, Exam 1 Solutions

1. Carefully define the following terms: $\binom{a}{b}$, floor, Commutativity theorem (for propositions), Distributivity theorem (for propositions).

The binomial coefficient is a function from pairs a, b of nonnegative integers, with $a \geq b$, to $\mathbb{N}$, given by $\frac{a!}{b!(a-b)!}$. Let $x \in \mathbb{R}$. The floor of x is the unique integer n that satisfies $n \leq x < n+1$. The Commutativity theorem states that for any propositions p, q, that $p \vee q \equiv q \vee p$ and $p \wedge q \equiv q \wedge p$. The Distributivity theorem states that for any propositions p, q, r, that $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ and $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.

2. Carefully define the following terms: Addition semantic theorem, Disjunctive Syllogism semantic theorem, contrapositive (proposition), predicate.

The Addition semantic theorem states that for any propositions p, q: $p \vdash p \vee q$. The Disjunctive Syllogism theorem states that for any propositions p, q: $(p \vee q), \neg p \vdash q$. The contrapositive of conditional proposition $p \rightarrow q$ is the proposition $(\neg q) \rightarrow (\neg p)$. A predicate is a collection of propositions, indexed by one or more free variables, each drawn from its domain.

3. Prove or disprove: For all $n \in \mathbb{N}$, $(n-1)! | (n+1)!$.

The statement is true. Applying the definition of factorial twice, we get $(n+1)! = n!(n+1) = (n-1)!(n)(n+1)$. Since $n(n+1)$ is an integer, applying the definition of "divides", we conclude that $(n-1)! | (n+1)!$.

4. Prove or disprove: For all odd a, b, $\frac{a+b}{2}$ is even.

The statement is false. To disprove, we need one specific counterexample, such as $a = 1, b = 1$, which are odd (since $1 = 2 \cdot 0 + 1$) for which $\frac{a+b}{2} = \frac{2}{2} = 1$, which is not even (since there is no integer we can multiply by 2 to get 1).

5. Let p, q be propositions. Prove or disprove: $(p \downarrow q) \rightarrow (p \uparrow q)$ is a tautology.

Because the fifth column of the truth table (to the right) is all T, the statement is a tautology.

p	q	$p \downarrow q$	$p \uparrow q$	$(p \downarrow q) \rightarrow (p \uparrow q)$
T	T	F	F	T
T	F	F	T	T
F	T	F	T	T
F	F	T	T	T

6. Without using truth tables, prove the Destructive Dilemma theorem, which states: Let p, q, r, s be arbitrary propositions. Then $p \rightarrow q, r \rightarrow s, (\neg q) \vee (\neg s) \vdash (\neg p) \vee (\neg r)$.

Due to the hypothesis that $(\neg q) \vee (\neg s)$, we consider two cases. If $\neg q$ is T, we combine this with $p \rightarrow q$ and modus tollens to conclude $\neg p$. By applying addition, $(\neg p) \vee (\neg r)$. If, instead, $\neg s$ is T, we combine this with $r \rightarrow s$ and modus tollens to conclude $\neg r$. By applying addition, $(\neg p) \vee (\neg r)$. In both cases, $(\neg p) \vee (\neg r)$.

7. Let $x \in \mathbb{R}$. Prove that if $2x \notin \mathbb{Q}$, then $3x + 1 \notin \mathbb{Q}$.

We use a contrapositive proof. Assume that $3x + 1 \in \mathbb{Q}$. Hence, there are $a, b \in \mathbb{Z}$ with $b \neq 0$ such that $3x + 1 = \frac{a}{b}$. We subtract one from each side to get $3x = \frac{a-b}{b}$. We then multiply both sides by $\frac{2}{3}$ to get $2x = \frac{2a-2b}{3b}$. Now, $2a - 2b, 3b$ are integers, and $3b \neq 0$, so $2x \in \mathbb{Q}$.

8. Let p, q, r, s be propositions. Simplify $\neg(((p \rightarrow q) \rightarrow r) \wedge s)$ as much as possible (where no compound propositions are negated).

We first apply De Morgan's law to get $(\neg((p \rightarrow q) \rightarrow r)) \vee (\neg s)$. Next we apply Theorem 2.16 (negated conditional interpretation) to get $((p \rightarrow q) \wedge (\neg r)) \vee (\neg s)$. Alternatively, we apply conditional interpretation to get $(\neg(\neg(p \rightarrow q) \vee r)) \vee (\neg s)$, then De Morgan's law to get $((\neg(\neg(p \rightarrow q))) \wedge (\neg r)) \vee (\neg s)$, then double negation to get $((p \rightarrow q) \wedge (\neg r)) \vee (\neg s)$.

9. Fix our domain to be $\mathbb{R}$. Simplify $\neg(\exists y\ \forall x\ \forall z\ (x < y) \rightarrow (x < z))$, as much as possible (where nothing is negated).

We first move the negation inward, getting $\forall y\ \exists x\ \exists z\ \neg((x < y) \rightarrow (x < z))$. Next we apply Theorem 2.16 (or conditional interpetation, De Morgan's law, and double negation) to get $\forall y\ \exists x\ \exists z\ (x < y) \wedge \neg(x < z) \equiv \forall y\ \exists x\ \exists z\ (x < y) \wedge (x \geq z)$.

10. Prove or disprove: $\exists x \in \mathbb{R}\ \forall y \in \mathbb{R},\ |y| \leq |y - x|$.

The statement is true. We need to find a specific example of x which will work for all y; the only one that works is $x = 0$. Now, let $y \in \mathbb{R}$ be arbitrary. $|y| = |y - 0| = |y - x|$, so $|y| \leq |y - x|$ is true.